
NOTES CONCERNING
THE YONEDA LEMMA

Let \mathcal{C} be a category. We write $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ for the category of contravariant functors $\mathcal{C} \rightarrow \mathbf{Set}$. This is also denoted as $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$, or *the category of presheaves over \mathcal{C}* .

We can define the Yoneda embedding, a functor $Y : \mathcal{C} \rightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ that takes:

- objects A of \mathcal{C} to the functor $\text{Hom}(-, A) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$
- arrows $f : A \rightarrow B$ to the natural transformation $\text{Hom}(-, f) : \text{Hom}(-, B) \Rightarrow \text{Hom}(-, A)$, which has components:

$$\begin{aligned} \text{Hom}(-, f)_C : \text{Hom}(C, B) &\rightarrow \text{Hom}(C, A) \\ (g : C \rightarrow B) &\mapsto f \circ g : C \rightarrow A \end{aligned}$$

For convenience, we may use $(f \circ -)$ to denote both the components and the natural transformation itself.

Note that for any C in \mathcal{C} , the hom-functor $\text{Hom}(-, C)$ sends arrows $f : A \rightarrow B$ to the function

$$\begin{aligned} \text{Hom}(f, C) : \text{Hom}(B, C) &\rightarrow \text{Hom}(A, C) \\ (g : B \rightarrow C) &\mapsto g \circ f \end{aligned}$$

which we may write as $(- \circ f)$, and is not to be confused with the *natural transformation* $(f \circ -)$ above.

Now, let $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ be a presheaf on \mathcal{C} .

Lemma 1 (Yoneda). *For every object C of \mathcal{C} there is a canonical isomorphism*

$$\text{Hom}(YC, F) \cong FC$$

Proof. Fix an object C of \mathcal{C} . Note that YC, F are themselves presheaves on \mathcal{C} , so the elements of the homset on the left are actually natural transformations. So we can equivalently write:

$$\text{Nat}(\text{Hom}(-, C), F) \cong FC.$$

We begin by defining a function

$$\begin{aligned}\varphi &: \text{Nat}(\text{Hom}(-, C), F) \rightarrow FC \\ \varphi(\eta) &= \eta_C(\text{id}_C)\end{aligned}$$

Next we verify that φ is injective. Suppose that $\varphi(\eta) = \varphi(\theta)$, so that

$$\eta_C(\text{id}) = \theta_C(\text{id}).$$

We want to show that $\eta_B = \theta_B : \text{Hom}(B, C) \rightarrow FB$ for any object B of \mathcal{C} . That is, that for any $f : B \rightarrow C$ we have $\eta_B(f) = \theta_B(f)$. So pick an arrow $f : B \rightarrow C$.

Now we form the naturality square of η :

$$\begin{array}{ccc}\text{Hom}(C, C) & \xrightarrow{(- \circ f)} & \text{Hom}(B, C) \\ \downarrow \eta_C & & \downarrow \eta_B \\ FC & \xrightarrow{Ff} & FB\end{array}$$

If we follow $\text{id}_C \in \text{Hom}(C, C)$ around both ways we see that:

$$\begin{aligned}\eta_B \circ (f \circ -)(\text{id}) &= Ff \circ \eta_C(\text{id}) \\ \eta_B(f) &= Ff(\eta_C(\text{id}))\end{aligned}$$

By forming the naturality square for θ we similarly see that

$$\begin{aligned}\theta_B(f) &= Ff(\theta_C(\text{id})) \\ &= Ff(\eta_C(\text{id})) \\ &= \eta_B(f)\end{aligned}$$

Therefore $\eta = \theta$, as desired, and we see that φ is indeed injective.

The final step is to show that φ is surjective. Pick an element x of FC . We want to find an η such that

$$\begin{aligned}\varphi(\eta) &= x \\ \eta_C(\text{id}) &= x\end{aligned}$$

We define η by

$$\eta_B(f) = Ff(x).$$

For if $f : B \rightarrow C$ then $Ff : FC \rightarrow FB$, and so $Ff(x)$ is an element of FB . So we have defined a family of arrows $\eta_B : \text{Hom}(B, C) \rightarrow FB$. It just remains to verify that η is natural.

Let $f : A \rightarrow B$, and form the naturality square:

$$\begin{array}{ccc}
\text{Hom}(B, C) & \xrightarrow{(- \circ f)} & \text{Hom}(A, C) \\
\downarrow \eta_B & & \downarrow \eta_A \\
FB & \xrightarrow{Ff} & FA
\end{array}$$

Pick a g in $\text{Hom}(B, C)$, so $g : B \rightarrow C$, and follow it around the square:

$$\begin{aligned}
Ff \circ \eta_B(g) &= Ff(Fg(x)) \\
&= F(g \circ f)(x) && \text{for } F \text{ is contravariant} \\
&= \eta_A(g \circ f) \\
&= \eta_A \circ (- \circ f)(g)
\end{aligned}$$

Hence we see that the square commutes and η is indeed natural. Therefore φ is surjective, and in fact bijective, which is what we wanted to show. \square

As a corollary, we see that

Proposition 2. *The Yoneda functor is in fact an embedding (fully faithful functor).*

Proof. We want to show that for every A, B of \mathcal{C} the map

$$\begin{aligned}
Y : \text{Hom}(A, B) &\rightarrow \text{Nat}(YA, YB) \\
(f : A \rightarrow B) &\mapsto (f \circ -) : YA \implies YB
\end{aligned}$$

is a bijection.

By above we have an iso:

$$\begin{aligned}
\varphi : \text{Nat}(YA, YB) &\rightarrow (YB)A \\
&= (\text{Hom}(-, B))A \\
&= \text{Hom}(A, B)
\end{aligned}$$

Suppose $f : A \rightarrow B$. Observe that:

$$\begin{aligned}
\varphi \circ Y(f) &= Yf(\text{id}) \\
&= (f \circ -)(\text{id}) \\
&= f
\end{aligned}$$

So we see that in fact $\varphi \circ Y = \text{id} : \text{Hom}(A, B) \rightarrow \text{Hom}(A, B)$. Therefore (as we know φ to be a bijection) it follows that Y is a bijection, as desired. \square